Steering the motion of rotary solitons in radial lattices

Y. J. He,1,2 Boris A. Malomed,3 and H. Z. Wang1,*

1State Key Laboratory of Optoelectronic Materials and Technologies, Zhongshan (Sun Yat-Sen) University, Guangzhou 510275, China
2School of Electronics and Information Engineering, Guangdong Polytechnic Normal University, Guangzhou 510665, China
3Department of Interdisciplinary Studies, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel

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We demonstrate that rotary motion of a two-dimensional soliton trapped in a Bessel lattice can be precisely controlled by application of a finite-time push to the lattice, due to the transfer of the lattice’s linear momentum to the orbital momentum of the soliton. A simple analytical consideration treating the soliton as a particle provides for an accurate explanation of numerical findings. Some effects beyond the quasi-particle approximation are explored too, such as destruction of the soliton by a hard push.

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I. INTRODUCTION

Techniques which make it possible to manipulate coherent matter-wave beams by means of specially designed potentials are crucial ingredients in many applications of atom optics, such as inertial sensors [1] and atomic holography [2]. In that context, an important topic is the control of rotational motion of Bose-Einstein condensates (BECs). Rotating BECs support Abrikosov lattices of quantized vortices [3], a counterpart of the fractional quantum Hall effect [4], persistent currents [5], and quantum phase transitions [6]. Rotating BECs can be trapped in toroidal magnetic [7] and optical waveguides [8]. Recently, the transfer of angular momentum from photons to atoms using a two-photon stimulated Raman process with Laguerre-Gaussian beams was demonstrated to generate a BEC vortex state [9].

Optical lattice (OL) is another versatile tool for the studies of dynamical properties of cold quantum gases [10]. Many fundamental effects in BECs loaded in OLs have been reported, including Bloch oscillations [11], Landau-Zener tunneling [12], and superfluid-insulator transitions [13]. Matter-wave solitons of the gap type were predicted [14] and created [10] in BEC loaded in an OL. It was also shown that motion of the OL relative to external trap can be used for the dynamical control of matter-wave solitons [15].

A promising application of the OLs is the stabilization of multidimensional solitons in BEC [16]. Rotating OLs are also available in the experiment [17]. It was predicted that 2D solitons can be stabilized by a rotating lattice in either of two forms: as a fully localized soliton trapped by a slowly revolving lattice and rotating in sync with it, at some distance from the rotation pivot, or a ring-shaped soliton or vortex, supported by a rapidly revolving OL [18]. Phase transitions of vortex matter in rotating lattices were predicted too [19]. In the limit of very rapid rotation, the potential of the revolving OL becomes equivalent to that of a radial lattice [18]. Previously, it had been demonstrated that such OLs of the Bessel type support stable 2D rotary solitons because they may perform circular motion in an annular potential trough (lattice ring) [20]. The same model supports stable 3D solitons [21]. If the nonlinearity is repulsive, the Bessel radial potential maintains ring solitons in the form of azimuthal dipoles and quadrupoles [22], and a potential periodic along the radius gives rise to radial gap solitons [23]. Experimentally, optical spatial solitons localized at the center or forming a ring, as well as rotating ones, were created in a cylindrical OL [24].

An essential issue is steering the motion of rotary solitons in the Bessel lattice. Here, we aim to demonstrate that 2D solitons trapped in a Bessel OL at a distance from the center can be set in rotation by pushing the lattice in a transverse direction, for a finite time (“jerking”). In other words, the OL’s linear momentum may be effectively converted into the orbital angular momentum of the rotary soliton. After the OL comes to a halt, the soliton keeps rotating due to inertia. This mechanism makes it possible to design a “motor” for accurate control of motion and transfer of rotary matter-wave solitons in radial lattices.

II. THE STATIONARY MODEL

The dynamics of the self-attractive BEC in radial potential \( V(r) \) is governed by the 2D Gross-Pitaevskii equation (GPE) for the mean-field wave function \( g(r,t) \). The scaled form of the equation is

\[
\frac{i}{\hbar} \frac{\partial g}{\partial t} = \left[ -\left(\frac{1}{2}\nabla^2 + V(r) - |q|^2 \right) g \right],
\]

(1)

where \( \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_r^2 \) with \( r^2 = x^2 + y^2 \). To cast the GPE in the form of Eq. (1), one rescales the wave function and other variable as follows: \( q \rightarrow q \sqrt{g_{2D}} \) with \( g_{2D} = 2 \sqrt{2} (|a_x|/a_L) \times (\omega_x/a_L) \), where \( a_x < 0 \) is the s-wave scattering length, \( \omega_z \) is the transverse trap frequency, the characteristic length is \( a_L = \pi \) (where \( e \) is the lattice period), and the characteristic frequency is \( \omega_0 = E_0/\hbar \). The recoil energy is \( E_0 = \hbar^2/(2m a_L^2) \) (m is the mass of the trapped atoms). The radial potential with the minimum at \( r = 0 \), corresponding to the Bessel OL, is

\[
V(r) = -p J_0(\sqrt{8(x^2 + y^2)}),
\]

(2)

where \( p > 0 \) is the strength of the lattice, its intrinsic scale was fixed using the invariance of Eq. (1), and \( J_0 \) is the Bessel
We consider the OL moving along the y axis at velocity \(-v\); to this end, \(y\) will be replaced by \(y+v t\).

First, axisymmetric solutions to Eq. (1) with the quiescent OL \((v=0)\), in the form of a soliton trapped in the central potential well, is looked for as \(q(r,t) = f(r) \exp(ibt)\), where \(-b\) is chemical potential, and \(f(r)\) is a real function obeying the equation \(f'' + r^{-1}f' -2[V(r) + b]f + 2f^3 = 0\), that can be solved by the shooting method.

The norm of the axisymmetric soliton solution, \(U = 2\pi \int_0^{\infty} f^2(r) dr\), is found to be a monotonically growing function of \(b\) [Fig. 1(a)], which implies the stability of the solitons according to the Vakhitov-Kolokolov criterion [25]. Similar to the situation known in the case of 2D solitons supported by the 2D or quasi-1D OL [16], the solitons exist not at all positive values of \(b\), but only at \(b > b_{CO} > 0\). The cutoff value \(b_{CO}\) grows with the OL strength [Fig. 1(b)], and the soliton’s norm vanishes at \(b = b_{CO}\) [Fig. 1(a)].

To find stationary solitons trapped in the circular trough corresponding to the first off-center minimum of radial potential \(2\), at \(r = r_{min} = 2.47\), a stable stationary soliton found in the central potential well was placed in the trough, and Eq. (1) was then directly integrated in time, demonstrating stable relaxation to the corresponding 2D soliton [Fig. 1(c)]. The established shape of the latter soliton was found to be identical to that which could be found from Eq. (1) by dint of the imaginary-time integration, while the present method provides for a faster convergence to the numerically accurate solution.

The stability of the axisymmetric solitons against azimuthal perturbations, which depend on angular variable \(\theta\), was examined in a rigorous form, by looking for a perturbed solution to Eq. (1) in the form of \(q = [f(r) + u(r)\exp(\delta t + i\phi) + w(r)\exp(\delta t - i\phi)]\exp(ibt)\), where \(u(r)\) and \(w(r)\) are eigenmodes of infinitesimal perturbations with integer azimuthal index \(n\) and (complex) instability growth rate \(\delta\). The linearization of Eq. (1) leads an eigenvalue problem, which was solved numerically. The stability condition \(\text{Re}(\delta) < 0\) was explicitly checked for \(n = 0, 1, 2, 3, 4, 5\), an instability at still higher values of \(n\) is very implausible (as also suggested by the VK criterion). The stability was also verified in direct simulations of the evolution of the solitons with added random perturbations, whose relative amplitude was set at the 10% level.

### III. THE MOVING LATTICE

Proceeding to the model with the radial OL suddenly set in motion at velocity \(-v\) along the y axis, we show in Fig. 2 that the static soliton placed in the annular trough [the same as in Fig. 1(c)], starts rotary motion. In this case, the push, with \(v = 1\), was applied perpendicular to the radial direction from the center to the initial location of the soliton, and the lattice stopped after having passed a short distance, \(d = 1.5\) (i.e., the lattice was subjected to a “jerk” during short time \(\tau = 1.5\)). An elementary kinematic consideration treating the soliton as a quasiparticle shows that, after the lattice comes to a halt at \(\tau = \pi\), the residual tangential velocity of the soliton is \(v_{res} = v[1 - \cos (\tau r_{min})] = 0.18\). Accordingly, the period of the resulting rotary motion is expected to be \(T = 2\pi r_{min}/v_{res} = 87\), which matches the numerical picture in Fig. 2. Taking typical values of physical parameters relevant to the experiment with the BEC of seven Li atoms, viz., transverse trapping frequency \(\omega_z = 2\pi \times 90\) Hz and number of atoms \(4 \times 10^5\), and undoing rescaling implied in the derivation of Eq. (1), we conclude that \(x = 1\), \(r = 1\), and \(v = 1\) correspond, in physical units, to 15 \(\mu\)m, 1.8 ms, and 15 mm/s, respectively.

In a more general situation, the push is applied to the OL with the soliton trapped at an azimuthal position \(\theta\) [Fig.
is readily explained by Eq. (3); indeed, in the range considered, \( d \leq 10 \) (recall \( r_{\text{min}} = 2.47 \)), it follows from Eq. (3) that \( v_{\text{res}} \) vanishes along curve \( d/(2r_{\text{min}}) = (\theta - \pi/2)/\sin \theta \) [red curve in Fig. 3(b)]. If the soliton comes to a halt after the application of the jerk, the finite shift of the soliton in the azimuthal direction can be found too, \( \Delta \theta = -(d/r_{\text{min}})\sin \theta \).

The above results were explained within the framework of the quasiparticle (adiabatic) approximation for the soliton. A nonadiabatic effect, which manifests the wave nature of the soliton, is the existence of a threshold value of the lattice displacement \( d_{\text{thr}} = 0.6 \): a very short jerk, with \( d < 0.6 \) does not set the soliton in progressive motion, generating only small oscillations about the initial position [domain D in Fig. 3(b)]. The threshold value depends on \( \theta \), attaining a minimum at \( \theta = \pi/4 \). Of course, the rotary motion cannot be generated by the application of the push in directions \( \theta = 0 \) and \( \theta = \pi \), i.e., \( d_{\text{thr}} \) diverges at these points, that is why the diagram in Fig. 3(b) is limited to range \( \pi/6 < \theta < 5\pi/6 \).

A more dramatic nonadiabatic effect is the destruction of the soliton if the velocity suddenly applied to the holding lattice exceeds a critical value \( v_{\text{max}} \), see Fig. 3(f) (in some cases, a part of the initial soliton’s norm may be kept by a smaller soliton which is pushed, along the radial direction, into the central potential well or the adjacent annular trough). A numerically found dependence of \( v_{\text{max}} \) on \( \theta \) is displayed in Fig. 3(c), the maximum of \( v_{\text{max}} \) at \( \theta = \pi/4 \) correlating with the above mentioned minimum of \( d_{\text{thr}} \) observed at the same point.

Finally, the application of the jerk can be used to accelerate, decelerate, reverse, and stop a rotary soliton which was originally moving at velocity \( v_0 \). In particular, a straightforward generalization of Eq. (3) makes it possible to predict the duration of the jerk (with velocity \( v \)), which is necessary to stop the soliton: 

\[
\tau = \frac{[\theta_0 - \sin^{-1}(\sin \theta_0 - v_0/v)]}{(\sin \theta_0)}
\]
\( -v_\theta /v \), where \( \theta_0 \) is the azimuthal position of the soliton at the moment of the application of the jerk. Examples of the above-mentioned outcomes are displayed in Fig. 4.

A possible generalization is to consider the motion of the rotary soliton driven by periodic jitter of the radial lattice, \( y(t) = a \sin(o t) \). In the adiabatic approximation, the respective equation of motion for the soliton in the reference frame attached to the lattice is \( r_{\text{min}} \theta = a o^2 \sin(\theta) \sin \theta \). It is well known that this equation may produce both regular and chaotic dynamical regimes. The same driving technique may be applied to predicted matter-wave solitons in toroidal traps [26].

IV. CONCLUSION

We demonstrate that a 2D matter-wave soliton trapped in the radial lattice can be set in controlled motion, or its motion can be altered as required, by jerking the holding lattice. The use of this “lattice motor” may avoid heating of the BEC implicated by other soliton-transport techniques, such as dragging by a focused laser beam. If the radial OL traps two solitons in the same annular trough, the jerk may be used to initiate collisions between them. It is also relevant to mention that, while the present model considered the BEC with attraction between atoms, in the case of repulsion the radial OL may trap stable multipole annular patterns. Due to their symmetry, the patterns cannot be set in motion by jerking. However, a more sophisticated approach may be possible in this case: first, the symmetry of the pattern can be broken by a sufficiently strong jerk, and then another jerk, in the perpendicular direction, will initiate the rotary motion of the deformed pattern.

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